

Quantum Field Theory of Spin Networks

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Abstract

We study the transition amplitudes in the state-sum models of quantum gravity in $D = 2, 3, 4$ spacetime dimensions by using the field theory over G^D formulation, where G is the relevant Lie group. By promoting the group theory Fourier modes into creation and annihilation operators we construct a Fock space for the quantum field theory whose Feynman diagrams give the transition amplitudes. By making products of the Fourier modes we construct operators and states representing the spin networks associated to triangulations of spatial boundaries of a triangulated spacetime manifold. The corresponding spin network amplitudes give the state-sum amplitudes for triangulated manifolds with boundaries. We also show that one can introduce a discrete time evolution operator, where the time is given by the number of D -simplices in a triangulation, or equivalently by the number of the vertices of the Feynman diagram. The corresponding transition amplitude is a finite sum of Feynman diagrams, and in this way one avoids the problem of infinite amplitudes caused by summing over all possible triangulations.

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1 Introduction

The idea of constructing a quantum theory of gravity by assuming that the spacetime is discrete at the Planck scale is appealing because it resolves by definition the problem of UV infinities of the conventional quantum field theory approach. However, the development of this idea was hindered by the complexity of the obvious candidate for such a theory, i.e. Regge simplicial formulation of general relativity (GR) [1].

In the past decade a new class of simplicial gravity models have been developed, which are based on the representation theory of the relevant symmetry group G . This group is $SO(D)$ for the Euclidean gravity case or $SO(D-1, 1)$ for the Lorentzian case. The prototype model was the Ponzano-Regge model of 3d gravity [2], which was made mathematically well defined in the work of Turaev and Viro by passing from the $SO(3)$ to the quantum $SU(2)$ group [3]. This was generalized to 4d in the approach of topological state sum models [4], which is a mathematical formalism for constructing topological invariants for manifolds out of colored triangulations, based on the category theory. In physics terms, this is a way of constructing partition functions for topological field theories. The state-sum formalism was based on the papers of Boulatov and Ooguri, who realized that the topologically invariant state sums can be generated as Feynman diagrams of a field theory over G^D [5, 6]. Their approach was motivated by the result from string theory that the partition function of a discretized string theory can be represented as a sum of Feynman diagrams of a matrix field theory (for a review and references see [7]).

Based on these developments, Barret and Crane have proposed an approach for obtaining the partition function for general relativity (GR) in $D = 4$, which is a non-topological theory, from a state sum of a topological theory, the BF theory [8, 9]. The BC model represents a very interesting formulation of discrete quantum gravity, since the local symmetry group of the spacetime plays a crucial role. One colors a triangulation \mathcal{T} of the spacetime manifold M with a certain class of irreducible representations (irreps) of the Lorentz group, and the partition function for a given \mathcal{T} is the sum over the amplitudes for all colorings. Given the partition function, one can compute the transition amplitude associated with a triangulation of M with spatial boundaries which are colored with a fixed set of irreps, by summing over the internal irreps, exactly as in the topological cases [2, 10, 11].

Since a field theory formulation of the BC model exists [12], such an amplitude would correspond to a Feynman diagram with external legs. In the quantum field theory (QFT) formalism such diagrams contribute to the matrix elements of an evolution operator, the S-matrix. This evolution operator acts in the Hilbert space of the QFT, which is the Fock space constructed out of the creation and the annihilation operators. The aim of this paper is to develop these concepts for the BC model.

A further motivation is to construct the states which correspond to the colored triangulations of the spatial boundaries, i.e. the spin network states. This is based on the property of the QFT formalism that one can associate states to operators via the Fock space vacuum state, so that one could in principle construct the operators corre-

sponding to spin networks. This would be also the first step for the third quantization formulation, i.e. a quantum theory where the number of the universes is not fixed.

Furthermore, one can define the transition amplitudes between these spin net states by summing over the appropriate Feynman diagrams with external legs. Since such a Feynman diagram (FD) corresponds to a triangulation of the spacetime with boundaries, there are several amplitudes one can associate to a transition from a state on the initial surface to a state on the final surface. The standard amplitude is given by the sum over all possible triangulations, or equivalently over all FD, but we argue here that it makes sense to consider the perturbative part of that amplitude, which is given by the sum of the FD with n vertices, or triangulations with n simplices, as the transition amplitude for the time interval of n Planck units of time. In this way one introduces a discrete time variable, and also regularizes the amplitude, since one avoids the infinite sum over all triangulations, which is a generically divergent expression.

In sections 2,3 and 4 we study the simplicial field theory for a 2d discretized gravity theory based on the 2d BF theory, since it is a good toy model for illustrating and developing our ideas. In section 5 we further develop and explore the general concepts introduced in the case of the 2d model, where we consider the 3d simplicial field theory of Boulatov. In sections 6 and 7 we apply these concepts to the field theory formulation of the BC model. In section 8 we present our conclusions.

2 D=2 model

We start with the $D = 2$ model, since it is very useful for understanding and illustrating the basic concepts. The D -dimensional BF theory is given by the action

$$S_{BF} = \int_M \langle B \wedge F \rangle \quad (1)$$

where B is a $(D - 2)$ -form, i.e. scalar field in 2d, and $F = dA + A \wedge A$ is the curvature two-form for the connection one-form A . A and B take values in the Lie algebra of the Lie group G and \langle, \rangle is the corresponding invariant bilinear form.

For the case of 2d gravity the relevant groups are the 2d Poincare group $ISO(1, 1)$, then the 2d anti-de-Sitter group $SO(1, 2)$ and its Euclidean versions $SO(3)$ and $SU(2)$. The corresponding BF theory gives the Jackiw-Taitelboim theory [13], which follows from the identifications

$$A = \omega J_0 + e^\pm J_\pm, \quad B = B^0 J_0 + B^\pm J_\pm \quad (2)$$

where ω is the spin connection, e^\pm are the zweibeins, B^0 is the dilaton, and J are the Lie algebra generators, so that

$$S_{BF} = \int_M B^0 (d\omega + \lambda_c e^+ \wedge e^-) + B^\mp (de^\pm \pm \omega e^\pm), \quad (3)$$

where λ_c is the cosmological constant. Hence the B^\pm enforce the zero-torsion conditions which give $\omega = \omega(e)$ and one ends up with the Jackiw-Teitelboim action. We will consider the compact G case, i.e. Euclidean 2d metrics, since the formulas are simpler, and the basic ideas are the same.

The field theory which generates the partition functions for the triangularizations of the two-manifold M as Feynman diagrams is given by

$$S_2 = \frac{1}{2} \int_{G^2} d^2g \varphi^2(g_1, g_2) + \frac{\lambda}{3!} \int_{G^3} d^3g \varphi(g_1, g_2) \varphi(g_2, g_3) \varphi(g_3, g_1) \quad , \quad (4)$$

where φ satisfies

$$\varphi(g_1, g_2) = \varphi(g_1 g, g_2 g) \quad , \quad (5)$$

and λ is the perturbation theory expansion parameter. Hence

$$\varphi(g_1, g_2) = \varphi(g_1 \cdot g_2^{-1}) = \sum_{\Lambda, \alpha, \beta} \phi_{\alpha\beta}^\Lambda D_{\alpha\beta}^\Lambda(g_1 \cdot g_2^{-1}) \quad , \quad (6)$$

where the last formula follows from the Peter-Weyl theorem. Λ denotes an unitary irreducible representation of G , $1 \leq \alpha, \beta \leq d_\Lambda = \dim \Lambda$. The Fourier coefficients ϕ^Λ will be important for the construction of the spin-network states and transition amplitudes, and they are the analogs of the creation and the annihilation operators from the particle field theory.

By inserting (6) into (4) and by using the orthonormality relations

$$\int_G dg (D_{\alpha\beta}^\Lambda(g))^* D_{\alpha'\beta'}^{\Lambda'}(g) = \frac{1}{d_\Lambda} \delta^{\Lambda, \Lambda'} \delta_{\alpha, \alpha'} \delta_{\beta, \beta'} \quad , \quad (7)$$

and the complex conjugation relations[†]

$$(D_{\alpha\beta}^\Lambda(g))^* = (-1)^{\Lambda - \alpha - \beta} D_{-\alpha - \beta}^\Lambda(g) \quad , \quad (8)$$

we obtain for the kinetic part of S

$$S_k = \frac{1}{2} \sum_{\Lambda, \alpha, \beta} d_\Lambda^{-1} (-1)^{\Lambda - \alpha - \beta} \phi_{\alpha\beta}^\Lambda \phi_{-\alpha - \beta}^\Lambda = \frac{1}{2} \sum_{\Lambda, \alpha, \beta} d_\Lambda^{-1} |\phi_{\alpha\beta}^\Lambda|^2 \quad , \quad (9)$$

while for the interaction part we get

$$S_v = \frac{\lambda}{3!} \sum_{\Lambda, \alpha, \beta, \gamma} d_\Lambda^{-2} \phi_{\alpha\beta}^\Lambda \phi_{\beta\gamma}^\Lambda \phi_{\gamma\alpha}^\Lambda \quad . \quad (10)$$

Note that (8) induces the following reality condition for the Fourier components ϕ

$$(\phi_{\alpha\beta}^\Lambda)^* = (-1)^{\Lambda - \alpha - \beta} \phi_{-\alpha - \beta}^\Lambda \quad , \quad (11)$$

[†]This form is for the $SU(2)$ case, so that $\Lambda = 2j = 0, 1, 2, \dots$ and $d_\Lambda = \Lambda + 1$.

which is a non-Abelian generalization of the reality condition for the usual Fourier modes: $a_p^* = a_{-p}$.

As far as the Feynmanology is concerned, it will be useful to consider a general ϕ^3 theory given by the action

$$S = \frac{1}{2} \sum_{I,J} \phi_I \phi_J C_{IJ} + \frac{\lambda}{3!} \sum_{I,J,K} \phi_I \phi_J \phi_K V_{IJK} \quad , \quad (12)$$

where the indices I, J, K belong to a general label set.

The perturbative expansion an the Feynman diagrams are generated by using the generating functional

$$Z[\mathcal{J}] = \int \mathcal{D}\phi \exp(i(S[\phi] + \sum_I \mathcal{J}_I \phi_I)) \quad , \quad (13)$$

where $\mathcal{D}\phi = \prod_I d\phi_I$. One can write

$$Z(\mathcal{J}) = \sum_I \mathcal{J}_{I_1} \cdots \mathcal{J}_{I_n} G_{I_1 \cdots I_n} \quad (14)$$

where the Green functions G can be written as

$$\begin{aligned} G_{I_1 \cdots I_n} &= \frac{i^n}{n!} \int \mathcal{D}\phi e^{i(S_k + S_v)} \phi_{I_1} \cdots \phi_{I_n} \\ &= \frac{i^n}{n!} \sum_m \frac{i^m}{m!} \int \mathcal{D}\phi e^{iS_k} (S_v)^m \phi_{I_1} \cdots \phi_{I_n} \\ &= \frac{i^n}{n!} \sum_m \frac{i^m}{m!} \langle (S_v)^m \phi_{I_1} \cdots \phi_{I_n} \rangle = \frac{i^n}{n!} \sum_m G_{I_1 \cdots I_n}^{(m)} \quad . \end{aligned} \quad (15)$$

The perturbative components $G^{(m)}$ can be calculated by a repeated differentiation of the Gaussian integral

$$Z_0[\mathcal{J}] = \int \mathcal{D}\phi \exp \left(iS_k[\phi] + i \sum_I \mathcal{J}_I \phi_I \right) = \Delta \exp \left(-i \frac{1}{2} \sum_{IJ} \mathcal{J}_I C_{IJ}^{-1} \mathcal{J}_J \right) \quad (16)$$

with respect to the currents \mathcal{J} , where Δ is a constant proportional to $\det^{-\frac{1}{2}} ||C_{IJ}||$. One can then show that the result for $G^{(m)}$ can be expressed by using the classical analog of Wick's theorem from quantum field theory, i.e. one makes all possible pairings in the string of ϕ 's given by $S_v \cdots S_v \phi_{I_1} \cdots \phi_{I_n}$ and then assigns to each pairing (ϕ_I, ϕ_J) a number equal to C_{IJ}^{-1} , and then takes the product of these numbers and sums over all possible sets of pairings.

Wick's theorem is the basis for the Feynman diagrammatic expansion since topologically non-equivalent sets of pairings can be denoted as trivalent graphs, which are the Feynman diagrams. For example, the term in $G_{IJKL}^{(2)}$ given by

$$\Gamma_{IJKL} = \# \sum_{I',J',K',L',M,N} C_{II'}^{-1} C_{JJ'}^{-1} C_{KK'}^{-1} C_{LL'}^{-1} C_{MN}^{-1} V_{I'J'M} V_{K'L'N} \quad (17)$$

where $\#$ is the number of equivalent pairings, will correspond to the H diagram. In order to obtain its value, which is the corresponding S-matrix contribution, or the transition amplitude, one has to remove the external legs propagator factors. This corresponds to calculating

$$\begin{aligned} A_{IJKL} &= \sum_{I'J'K'L'} C_{II'} C_{JJ'} C_{KK'} C_{LL'} \Gamma_{I'J'K'L'} \\ &= \# \sum_{MN} C_{MN}^{-1} V_{IJM} V_{KLN} \quad . \end{aligned} \quad (18)$$

The partition function is given by $\langle e^{iS_v} \rangle$ and it is equal to the sum of all vacuum diagrams. The connected diagrams are generated by $W[\mathcal{J}] = \log Z[\mathcal{J}]$. If $W = \sum_n W_n \lambda^n$, then the connected vacuum diagrams are given by

$$\begin{aligned} W_2 &= (\text{theta}) + (\text{dumbbell}) \\ W_4 &= (\text{Mercedes} - \text{Benz}) + (\text{cylinder}) + \dots \\ W_6 &= (\text{prism}) + \dots \quad , \end{aligned} \quad (19)$$

where

$$\begin{aligned} (\text{theta}) &= \# V_{IJK} V_{LMN} C^{IL} C^{JM} C^{KN} \\ (\text{dumbbell}) &= \# V_{IJK} V_{LMN} C^{IJ} C^{KL} C^{MN} \\ (\text{Merc.} - \text{Benz}) &= \# V_{IJK} V_{LMN} V_{OPQ} V_{RST} C^{ON} C^{PR} C^{QK} C^{LI} C^{JS} C^{TL} \quad , \end{aligned} \quad (20)$$

and so on, where $C^{IJ} = C_{IJ}^{-1}$ and $\#$ is the corresponding combinatorial factor.

Hence the vacuum Feynman diagrams denote the rules of forming invariants by contracting the indices in the product of tensor quantities V , i.e. given a product $V_{I_1 J_1 K_1} \dots V_{I_n J_n K_n}$, the corresponding Feynman diagrams represent all topologically non-equivalent ways of contracting the indices in this product with $C^{IJ} = C_{IJ}^{-1}$. The matrix C^{IJ} is the propagator in the field theory language. If a set of indices $\{I, J, \dots\}$ remains uncontracted, then this gives a Feynman diagram with external legs, which contributes to the transition amplitude for a set of states carrying the quantum numbers $\{I, J, \dots\}$.

Consider the $(n + m)$ -point Green function

$$G_{n+m} = \frac{i^{n+m}}{(n+m)!} \langle e^{iS_v} \phi_{I_1} \dots \phi_{I_n} \phi_{J_1} \dots \phi_{J_m} \rangle . \quad (21)$$

It will generate all FD with $(n + m)$ external legs. Note that the subclass of these FD containing all connected FD which have a string[‡] of n legs and a string of m legs will contribute to the amplitude for a transition from a state labeled with quantum numbers I_1, \dots, I_n to the state labeled with quantum numbers J_1, \dots, J_m . This can be

[‡]By a string we mean that there is no vertex with an internal line between the legs of a string.

explicitly realized by using the concept of creation and annihilation operators from the operator quantization formalism. The index label set I can be always split as $I = \{i, -i, j, -j, k, l\}$ such that

$$\phi_i^* = \phi_{-i} \quad , \quad \phi_j^* = -\phi_{-j} \quad , \quad \phi_k^* = \phi_k \quad , \quad \phi_l^* = -\phi_l \quad . \quad (22)$$

Then promote $\phi_{\pm i}$ and $\phi_{\pm j}$ modes into operators $\phi_{\pm i}$ and $\sqrt{-1}\phi_{\pm j}$ satisfying

$$\phi_i^\dagger = \phi_{-\tilde{i}} \quad , \quad [\phi_i, \phi_j^\dagger] = \delta_{\tilde{i}, \tilde{j}} \quad , \quad (23)$$

where $\tilde{i}, \tilde{j} \in \{i, j\}$. The zero-modes ϕ_k and ϕ_l promote into operators $\frac{1}{2}(\phi_k + \phi_k^\dagger)$ and $\frac{1}{2}(\phi_l - \phi_l^\dagger)$ where

$$[\phi_{\tilde{k}}, \phi_{\tilde{l}}^\dagger] = \delta_{\tilde{k}, \tilde{l}} \quad , \quad , \quad \tilde{k}, \tilde{l} \in \{k, l\} \quad , \quad (24)$$

and all other commutators are zero.

These operators naturally act on the Hilbert space of states

$$\phi_{i'_1}^\dagger \cdots \phi_{i'_n}^\dagger |0\rangle \quad , \quad \phi_{i'} |0\rangle = 0 \quad , \quad (25)$$

where now $i' \in \{i, j, k, l\}$. Then the amplitude for a transition from the initial state

$$|\Psi_1\rangle = \phi_{i'_1}^\dagger \cdots \phi_{i'_n}^\dagger |0\rangle \quad (26)$$

to the final state

$$|\Psi_2\rangle = \phi_{j'_1}^\dagger \cdots \phi_{j'_m}^\dagger |0\rangle \quad , \quad (27)$$

is given by

$$A_{12} = \langle 0 | \phi_{j'_1} \cdots \phi_{j'_m} \exp(i : S_v :) \phi_{i'_1}^\dagger \cdots \phi_{i'_n}^\dagger |0\rangle \quad , \quad (28)$$

where $: S_v :$ is the normal-ordered operator S_v with respect to the vacuum $|0\rangle$, and all the subsequent manipulations are the same as in the particle field theory case, i.e. one uses the operator formulation of Wick's theorem.

Note that one does not need to go to the operator formalism in order to compute the amplitudes. One can simply say that there are abstract states $|I_1 \dots I_n\rangle$ and the transition amplitude from the state $|J_1 \cdots K_1\rangle$ to the state $|L_2 \cdots M_2\rangle$ will be given by the sum of the corresponding Feynman diagrams. We will denote the result as

$$A_{12} = \langle L_2 \cdots M_2 | \exp(i S_v) | J_1 \cdots K_1 \rangle \quad . \quad (29)$$

We will also use the notation

$$A_{12} = \langle \phi_{L_2} \cdots \phi_{M_2} \exp(i S_v) \phi_{J_1} \cdots \phi_{K_1} \rangle \quad , \quad (30)$$

because it indicates that the transition amplitude is obtained from the Green's function by taking only the contractions corresponding to connected Feynman diagrams with a string of n and a string of m external legs, which we will denote as (n, m) FD.

The expression (30) is calculated perturbatively in λ , where at order n one has to calculate

$$A_{12}^{(n)} = \langle \phi_{L_2} \cdots \phi_{M_2} (S_v)^n \phi_{J_1} \cdots \phi_{K_1} \rangle \quad . \quad (31)$$

In the case of field theories of particles, the physically relevant quantity is (29), which gives the transition amplitude from a state in a distant (infinite) past to a state in a distant (infinite) future. Therefore in order to obtain meaningful amplitudes one has to sum all Feynman diagrams up to a given order of perturbation theory. However, in the case of field theories associated to simplicial gravity, we will argue that even the individual Feynman diagrams, or certain sub-classes of diagrams, can have a physical interpretation. This means that it is not necessary to sum over all possible triangulations interpolating between the initial and the final surface.

Note for example that in the case of the vacuum diagrams, a single diagram Γ is the partition function for the simplicial complex \mathcal{T} where $\Gamma = \mathcal{T}_1^*$ is the dual one-complex to \mathcal{T} . If one thinks about the simplicial gravity as the fundamental theory, i.e. not as an auxiliary device for defining the path integral of the continuum theory, then each equivalence class of simplicial complexes could have a physical meaning. This is trivially satisfied in the case of topological theories, where due to the triangulation independence, the continuum theory is the same as the simplicial theory, and hence the partition function for a single simplicial complex $\mathcal{T}(M)$ is the partition function for M .

3 Partition function

In order to develop these ideas further, we examine in some detail our 2d simplicial gravity model. Consider the Mercedes-Benz (MB) diagram. It is a dual one-complex for a tetrahedron t , if the t is considered as a 2d simplicial complex. In that case $t = \mathcal{T}(S^2)$, and hence the value of the MB diagram will be the partition function for a sphere S^2 . The kinetic term (9) implies that the propagator will be given by two parallel lines, representing the basic contraction

$$\langle \phi_{\alpha\beta}^\Lambda \phi_{\alpha'\beta'}^{\Lambda'} \rangle = (-1)^{\Lambda-\alpha-\beta} d_\Lambda \delta^{\Lambda,\Lambda'} \delta_{\alpha,-\alpha'} \delta_{\beta,-\beta'} \quad . \quad (32)$$

The interaction term (10) implies that the vertex will be given by a triangle whose corners are cut, where the indices $\alpha_i \in \Lambda_i$ are assigned to each edge. One then joins the vertices by the propagator lines, and in this way one obtains a ribbon MB graph. The value of the diagram will be given by the product of the edge delta functions multiplied by the propagator factors and the vertex factors. In general there will be $2E$ delta functions, which will be multiplied by a factor $(-1)^{E\Lambda} (d_\Lambda)^{E-2V}$, where E is the number of edges and V is the number of vertices of Γ . For the MB case $E = 6$ and $V = 4$. By summing over the indices along a loop of Γ , the corresponding delta functions give the factor d_Λ , so that

$$Z(\Gamma) = \sum_{\Lambda} (d_\Lambda)^{E-2V+F} (-1)^{E\Lambda}$$

$$= \sum_{\Lambda} (d_{\Lambda})^{V-E+F} (-1)^{E\Lambda} = \sum_{\Lambda} (d_{\Lambda})^{\chi(M)} (-1)^{E\Lambda} \quad , \quad (33)$$

where F is the number of loops in the ribbon graph Γ ($F(MB) = 4$), which is the same as the number of faces in \mathcal{T}^* , and we have used that $2E = 3V$ for trivalent graphs.

Since $\chi(\Gamma) = \chi(\mathcal{T}(M)) = \chi(M) = 2 - 2g$, the Euler number, is a topological invariant, where g is the genus of M , Z would be a topological invariant provided that there is no sign factor $(-1)^{E\Lambda}$. It comes from the sign factor in the propagator (32). There are several ways one can get rid off this sign factor. One can change the kinetic term into

$$\tilde{S}_k = \frac{1}{2} \sum_{\Lambda, \alpha, \beta} d_{\Lambda}^{-1} \phi_{\alpha\beta}^{\Lambda} \phi_{-\alpha-\beta}^{\Lambda} \quad , \quad (34)$$

so that the sign factor is removed from the propagator. Although this choice gives the correct answer, this action does not follow in general from the integration over the group, unless one has a group where one can choose a basis where $(D_{\alpha\beta}^{\Lambda})^*(g) = D_{-\alpha-\beta}^{\Lambda}(g)$. Similarly, one can choose

$$\tilde{S}_k = \frac{1}{2} \sum_{\Lambda, \alpha, \beta} d_{\Lambda}^{-1} (\phi_{\alpha\beta}^{\Lambda})^2 \quad , \quad (35)$$

and this would follow from the group integration if the group G allows a basis where the matrices $D(g)$ are real. In both cases we will write the propagator as

$$\langle \phi(12) \phi(34) \rangle = \delta(13) \delta(24) \quad . \quad (36)$$

One can avoid the propagator factor d_{Λ}^{-1} by rescaling ϕ , which then changes the vertex factor to $d_{\Lambda}^{-\frac{1}{2}}$. Either way the propagator (36) gives the required result

$$Z(M) = \sum_{\Lambda} (d_{\Lambda})^{\chi(M)} \quad . \quad (37)$$

If one wants to have an integral over the group expression which is valid for arbitrary G , then one can choose

$$S_k = \frac{1}{2} \int_{G^2} d^2g \varphi(g_1, g_2) \varphi(g_2, g_1) = \frac{1}{2} \sum_{\Lambda, \alpha, \beta} \phi_{\alpha\beta}^{\Lambda} \phi_{\beta\alpha}^{\Lambda} \quad . \quad (38)$$

This choice gives a twisted propagator

$$\langle \phi(12) \phi(34) \rangle = \delta(14) \delta(23) \quad . \quad (39)$$

The difference between the twisted propagator case (39) and the untwisted case (36) is that the same graph will not describe the same surface in each case. For example, in the case of the theta graph, the untwisted propagator gives $\chi = 2$, a sphere, while the

twisted propagator gives $\chi = 0$, a torus. In the untwisted case the torus is obtained by braiding two edges of the theta graph, which for the twisted case gives a sphere.

Note that one can impose the symmetry condition $\varphi(1, 2) = \varphi(2, 1)$, which together with the reality condition implies

$$(\phi_{\alpha\beta}^\Lambda)^* = \phi_{\beta\alpha}^\Lambda \quad , \quad (40)$$

i.e. ϕ^Λ are hermitian matrices. This gives a group theory generalization of the hermitian matrix models used in string theory.

4 Transition amplitudes and spin networks

Now let us try to formulate a transition amplitude from a spatial section Σ_1 to a spatial section Σ_2 , which are boundaries of M , i.e. we cut two holes in M . Consider the case when the corresponding simplicial complex is a prism. Then the base triangles correspond to triangulations of $\Sigma_i = S^1$, where S^1 is a circle. Each side of the prism is divided by a diagonal into two triangles, and this defines our triangulation $\mathcal{T}(M)$ where $M = S^2$ a two-sphere. The corresponding Feynman diagram is given by the dual graph, which is a circle with six legs. Its value can be obtained by making appropriate contractions in

$$\langle \phi_{\alpha_1\beta_1}^{\Lambda_1} \phi_{\alpha_2\beta_2}^{\Lambda_2} \phi_{\alpha_3\beta_3}^{\Lambda_3} (S_v)^6 \phi_{\alpha_4\beta_4}^{\Lambda_4} \phi_{\alpha_5\beta_5}^{\Lambda_5} \phi_{\alpha_6\beta_6}^{\Lambda_6} \rangle \quad . \quad (41)$$

If we take (41) as it is, we will obtain an expression which is not G invariant, because of the uncontracted representation indices. In order to remedy this, consider the following quantity, or the operator

$$\Phi_3 = \int_{G^3} d^3g \varphi(g_1, g_2) \varphi(g_2, g_3) \varphi(g_3, g_1) \quad . \quad (42)$$

It is proportional to S_v , so if we use (10), we get

$$\Phi_3 = \sum_{\Lambda, \alpha, \beta, \gamma} d_\Lambda^{-2} \phi_{\alpha\beta}^\Lambda \phi_{\beta\gamma}^\Lambda \phi_{\gamma\alpha}^\Lambda = \sum_{\Lambda} d_\Lambda^{-2} \Phi_3(\Lambda) \quad . \quad (43)$$

We then propose that $\Phi_3(\Lambda)$ be an operator associated to a 1d spin network given by a triangle whose each side is colored by the irrep Λ . Note that $\Phi_3(\Lambda)$ is defined up to a scale factor, so we can write

$$\Phi_3(\Lambda) = \frac{1}{N(\Lambda)} \sum_{\alpha, \beta, \gamma} \phi_{\alpha\beta}^\Lambda \phi_{\beta\gamma}^\Lambda \phi_{\gamma\alpha}^\Lambda = \frac{1}{N(\Lambda)} \text{Tr}(\phi^\Lambda)^3 \quad . \quad (44)$$

A natural normalization factor would be $N(\Lambda) = d_\Lambda^{3/2}$.

One can then associate states $|\Phi_3(\Lambda)\rangle$ to operators $\Phi_3(\Lambda)$, and the corresponding amplitude will be given by

$$\begin{aligned} A_{33}^{(6)} &= \langle \Phi_3(\Lambda) | (S_v)^6 | \Phi_3(\Lambda) \rangle = \langle \Phi_3(\Lambda) (S_v)^6 \Phi_3(\Lambda) \rangle \\ &= \frac{1}{N^2(\Lambda)} \sum_{\alpha\beta\gamma} \langle \phi_{\alpha_1\beta_1}^\Lambda \phi_{\beta_1\gamma_1}^\Lambda \phi_{\gamma_1\alpha_1}^\Lambda (S_v)^6 \phi_{\alpha_2\beta_2}^\Lambda \phi_{\beta_2\gamma_2}^\Lambda \phi_{\gamma_2\alpha_2}^\Lambda \rangle \quad . \end{aligned} \quad (45)$$

If we think of Ψ_3 as a quantity associated to a triangle whose each vertex carries a group element g , then $\Phi_3(\Lambda)$ can be interpreted as a quantity corresponding to a spin net formed by a spatially dual diagram to the triangle, where the duality is with respect to a 1d space, and hence one obtains a triangle as a dual complex. Note that we can think of $\varphi(g_1, g_2) = \varphi(1, 2)$ as a quantity associated to the edge at whose ends are attached the group elements g_1 and g_2 . Therefore the quantity

$$\Phi_n = \int_{G^n} d^n g \varphi(1, 2) \varphi(2, 3) \cdots \varphi(n, 1) \quad (46)$$

will correspond to an n -sided polygon, whose each vertex carries a group element. Hence Φ_n will generate a spin network variable $\Phi_n(\Lambda) \propto Tr(\phi^\Lambda)^n$, which will correspond to a closed 1d spin network with n edges colored by Λ .

It is easy to see that the non-zero transition amplitudes will come from the perturbative amplitudes

$$A_{lm}^{(k)} = \langle \Phi_m(\Lambda) | (S_v)^k | \Phi_l(\Lambda) \rangle = \langle \Phi_m(\Lambda) (S_v)^k \Phi_l(\Lambda) \rangle \quad . \quad (47)$$

5 Time variables

Given this setup, and our remarks at the end of section 2, we can say something about possible time intervals for the amplitudes. The amplitude for the prism (45) could be interpreted as a transition amplitude from the state Φ_1 to the state Φ_2 in one unit of discrete time, so that the transition amplitude in n units of time would be given by a sum of FD corresponding to triangulations consisting of n prisms put on top of each other. This suggests a discrete time variable T to be the minimal distance, i.e. the number of edges, from the initial to the final triangle in the corresponding triangulation. However, this time variable will make sense only for triangulations where the minimal distance between any two points of the initial and the final triangle does not exceed $n + 1$. In general case these are the triangulations where for every point p of the initial polygon P we have

$$\min \{d(p, p') \mid p' \in P'\} = n \quad , \quad (48)$$

where P' is the final polygon.

Note that the triangulations which satisfy (48) can be sliced into $(n - 1)$ polygons P_k , such that the distance (48) is one for each pair (P_k, P_{k+1}) , where $P_0 = P$ and $P_n = P'$. It is easy to see that there will be infinitely many such triangulations for

every $n \geq 2$. Hence if one tries to define the amplitude $A(n)$ for n units of time as a sum over triangulations with $n - 1$ slices, then for $n \geq 2$ the amplitude $A(n)$ will contain infinitely many FD. Therefore one will have the same problem as in the case of the unrestricted amplitude

$$A_{lm} = \langle \Phi_m(\Lambda) \exp(iS_v) \Phi_l(\Lambda) \rangle \quad . \quad (49)$$

One can then try to restrict the infinite sum of FD in $A(n)$ by using the perturbation theory, i.e. by taking only the FD up to a given order of λ . However, one has to insure that the corresponding amplitudes are consistent. Namely, the amplitudes have to obey the composition rules, which come from the procedure of pasting two cylinders. This is equivalent to requiring that the amplitudes are given by the matrix elements of an evolution operator $U(T)$, which satisfies

$$U(T_1)U(T_2) = U(T_1 + T_2) \quad , \quad (50)$$

where T is the evolution parameter, i.e. the time variable.

For example, in the case of particle field theories, one has $U(T) = \exp(iS_v) = \exp(iTH_v)$, where the split $S_v = TH_v$ is possible because of the fixed spacetime background structure. In the case of simplicial field theories, this split is not natural, and it is not clear how to do it. One can then use the minimal distance time variable T , and the corresponding $U(T)$ would be defined by requiring that its matrix elements are given by the restricted amplitudes $A(n)$. However, it is not clear why would such an operator satisfy (50), and it is difficult to check this.

On the other hand, note that $U(T) = (S_v)^T$ satisfies the composition law (50). Hence the amplitude (47) can be interpreted as the transition amplitude for $T = k$ units of time, where the time variable is given by the number of vertices in the corresponding Feynman diagrams, or equivalently, by the number of triangles in the corresponding triangulations. Since the amplitude (47) is given by a finite sum of FD, the new evolution operator will give finite amplitudes for finite time intervals, provided that the individual FD are finite. This can be achieved by going to the quantum group formulation.

Since this choice of time is related to the covariant quantization procedure we are using, i.e. we are not splitting the spacetime into space and time, as in the canonical quantization, we will call this evolution parameter the covariant time.

However, there is a peculiarity with the covariant evolution operator. Since it is natural to define S_v to be a hermitian operator in the field theory Fock space, so that e^{iS_v} is a unitary operator, then $U(n) = (S_v)^n$ is going to be only a hermitian operator, but not a unitary operator. This would mean a non-unitary time evolution of states. However, one has to keep in mind that we are not dealing here with the usual domain of quantum mechanics, i.e. objects in a fixed background spacetime. We have here states describing the whole universe, which is in our toy model represented by a colored triangulation of S^1 , and hence one cannot apriori expect that the same rules apply as in the standard quantum mechanics.

6 D=3 model

We now examine our constructions for the case of 3d gravity. The partition function for the 3d BF theory is generated by the FD of a ϕ^4 field theory on G^3 [5], whose action is given by

$$S_3 = \frac{1}{2} \int_{G^3} d^3g \varphi^2(1, 2, 3) + \frac{\lambda}{4!} \int_{G^6} d^6g \varphi(1, 2, 3) \varphi(1, 4, 5) \varphi(2, 5, 6) \varphi(3, 6, 4) \quad , \quad (51)$$

where we use a short-hand $\varphi(1, 2, 3) = \varphi(g_1, g_2, g_3)$ and φ satisfies

$$\varphi(g_1, g_2, g_3) = \varphi(g_1g, g_2g, g_3g) \quad . \quad (52)$$

The condition (52) implies

$$\varphi(g_1, g_2, g_3) = \int_G dg \phi(g_1g, g_2g, g_3g) \quad , \quad (53)$$

where ϕ is an unconstrained field.

Note that the form of S_3 allows the following (category theory) interpretation. If we associate $\varphi(1, 2, 3)$ to a triangle whose edges are labeled with the group elements g_1, g_2, g_3 , the interaction term S_v can be associated to a tetrahedron whose edges are labeled by the group elements g_1, \dots, g_6 , and the triangles have orientations provided by the outer normals. With this interpretation it is natural to require the cyclic symmetry for φ , i.e.

$$\varphi(1, 2, 3) = \varphi(2, 3, 1) = \varphi(3, 1, 2) \quad . \quad (54)$$

However, as in the 2d case, one does not have to impose this symmetry.

The Fourier modes expansion of (53) gives

$$\varphi = \sum_{\Lambda, \alpha} \sqrt{d_{\Lambda_1} d_{\Lambda_2} d_{\Lambda_3}} \phi_{\alpha_1 \alpha_2 \alpha_3}^{\Lambda_1 \Lambda_2 \Lambda_3} D_{\alpha_1 \alpha_2 \alpha_3}^{\Lambda_1 \Lambda_2 \Lambda_3}(g_1, g_2, g_3) \quad , \quad (55)$$

where

$$\phi_{\alpha_1 \alpha_2 \alpha_3}^{\Lambda_1 \Lambda_2 \Lambda_3} = \sum_{\beta} \phi_{\alpha_1 \beta_1 \alpha_2 \beta_2 \alpha_3 \beta_3}^{\Lambda_1 \Lambda_2 \Lambda_3} \bar{C}_{\beta_1 \beta_2 \beta_3}^{\Lambda_1 \Lambda_2 \Lambda_3} \quad , \quad (56)$$

$$D_{\alpha_1 \alpha_2 \alpha_3}^{\Lambda_1 \Lambda_2 \Lambda_3}(g_1, g_2, g_3) = \sum_{\beta} D_{\alpha_1 \beta_1}^{\Lambda_1}(g_1) D_{\alpha_2 \beta_2}^{\Lambda_2}(g_2) D_{\alpha_3 \beta_3}^{\Lambda_3}(g_3) C_{\beta_1 \beta_2 \beta_3}^{\Lambda_1 \Lambda_2 \Lambda_3} \quad . \quad (57)$$

The coefficients C are basis components of the intertwiner map ι_3

$$\iota_3 : V(\Lambda_1) \otimes V(\Lambda_2) \otimes V(\Lambda_3) \rightarrow V(0) \quad , \quad (58)$$

where $V(0)$ is the subspace of the singlets, so that

$$|0; \Lambda_1, \Lambda_2, \Lambda_3\rangle = \sum_{\alpha} C_{\alpha_1 \alpha_2 \alpha_3}^{\Lambda_1 \Lambda_2 \Lambda_3} |\alpha_1\rangle \otimes |\alpha_2\rangle \otimes |\alpha_3\rangle \quad , \quad (59)$$

and $|\alpha\rangle$ are vectors of an orthonormal basis.

In order to obtain (55) one needs the relation

$$\int_G dg U^{\Lambda_1}(g) \otimes U^{\Lambda_2}(g) \otimes U^{\Lambda_3}(g) = \iota_3 \iota_3^\dagger, \quad (60)$$

which in the basis (59) becomes

$$\int_G dg D_{\alpha_1 \beta_1}^{\Lambda_1}(g) D_{\alpha_2 \beta_2}^{\Lambda_2}(g) D_{\alpha_3 \beta_3}^{\Lambda_3}(g) = C_{\alpha_1 \alpha_2 \alpha_3}^{\Lambda_1 \Lambda_2 \Lambda_3} \bar{C}_{\beta_1 \beta_2 \beta_3}^{\Lambda_1 \Lambda_2 \Lambda_3}. \quad (61)$$

By inserting (55) into S_3 we obtain

$$S_k = \frac{1}{2} \sum_{\Lambda, \alpha} \left| \phi_{\alpha(123)}^{\Lambda(123)} \right|^2, \quad (62)$$

and

$$S_v = \frac{\lambda}{4!} \sum_{\Lambda, \alpha} (-1)^{\sum_i (j_i - \alpha_i)} \phi_{\alpha(123)}^{\Lambda(123)} \phi_{\alpha(-145)}^{\Lambda(145)} \phi_{\alpha(-2-56)}^{\Lambda(256)} \phi_{\alpha(-3-6-4)}^{\Lambda(364)} \left\{ \begin{matrix} \Lambda_1 & \Lambda_2 & \Lambda_3 \\ \Lambda_4 & \Lambda_5 & \Lambda_6 \end{matrix} \right\}, \quad (63)$$

where we have used a shorthand $X(jkl) = X_j X_k X_l$ and

$$\left\{ \begin{matrix} \Lambda_1 & \Lambda_2 & \Lambda_3 \\ \Lambda_4 & \Lambda_5 & \Lambda_6 \end{matrix} \right\} = \sum_{\beta} (-1)^{\sum_i (j_i - \beta_i)} C_{\beta(123)}^{\Lambda(123)} C_{\beta(-145)}^{\Lambda(145)} C_{\beta(-2-56)}^{\Lambda(256)} C_{\beta(-3-6-4)}^{\Lambda(364)} \quad (64)$$

is the $6j$ symbol. We also denote it as $6j(\Lambda_1 \cdots \Lambda_6)$.

Since the reality condition can be written as

$$\left(\phi_{\alpha(123)}^{\Lambda(123)} \right)^* = (-1)^{\sum_i (j_i - \alpha_i)} \phi_{\alpha(-1-2-3)}^{\Lambda(123)}, \quad (65)$$

then the kinetic term (62) gives for the propagator

$$\langle \phi_{\alpha(123)}^{\Lambda(123)} \phi_{\alpha'(123)}^{\Lambda'(123)} \rangle = \prod_{i=1}^3 (-1)^{j_i - \alpha_i} \delta^{\Lambda_i, \Lambda'_i} \delta_{\alpha_i, -\alpha'_i}. \quad (66)$$

Note that the 3d propagator contains a sign factor analogous to the one in the 2d case. Since for our purposes this sign factor, as well as the related sign factors, will not be important, we will omit them from the formulas. We will then write the formulas as for the real $D(g)$ case.

The Feynman diagrams will be given by the four-valent graphs Γ , where each edge is represented by 3 parallel lines, while each vertex is represented by a tetrahedron whose corners are cut and the corresponding edges are joined with the propagator lines. The Feynman rules which follow from (62) and (63) consist of assigning an index $\alpha_i \in \Lambda_i$ to each edge of each tetrahedron, as well as the corresponding $6j$

symbol. One then assigns a delta function for each propagator line and multiplies all the delta functions with the $6j$ symbols. If Γ has $|V|$ vertices and $|E|$ edges, one then obtains a product of $|V|$ $6j$ symbols and $3|E|$ delta functions. By summing over the indices for each loop[§] F of Γ one obtains a $d(\Lambda_F)$ factor, so that the value of the diagram is given by

$$Z(\Gamma) = \sum_{\Lambda_F} \prod_{F \in \Gamma} d(\Lambda_F) \prod_{V \in \Gamma} 6j(\Lambda_{F_1(V)} \cdots \Lambda_{F_6(V)}) . \quad (67)$$

For example, consider a vacuum diagram given by the pentagram graph. The pentagram is a dual one-complex to the 4-simplex if the 4-simplex is considered as a 3d simplicial complex. This 4-simplex can be considered as a triangulation of the space-time manifold S^3 .

If the graph Γ is dual to a triangulation $\mathcal{T}(M)$, we can rewrite Z as

$$Z(M) = \sum_{\Lambda_e} \prod_{e \in \mathcal{T}} d(\Lambda_e) \prod_{t \in \mathcal{T}} 6j(\Lambda_{e_1(t)} \cdots \Lambda_{e_6(t)}) \quad (68)$$

where e are the edges of \mathcal{T} , which correspond to the faces F of Γ . t are the tetrahedrons of \mathcal{T} , which correspond to the vertices V of Γ . In this way one reproduces the Ponzano-Regge formula, where the edges e of the triangulation \mathcal{T} are colored with the $SU(2)$ irreps, and the $6j$ symbols are assigned to the tetrahedrons t . The sum over the colorings is divergent, and it can be regularized by using the quantum group $SU_q(2)$ [3]. The result is triangulation independent, so that Z_q is a topological invariant of M .

As far as the transition amplitudes are concerned, they will be given by a straightforward generalization of the $D = 2$ amplitudes. For example, if we wish to calculate the transition amplitude from an initial surface $\Sigma_1 = S^2$ to a final surface $\Sigma_2 = S^2$, we take a 3d simplicial complex with two disjoint boundary tetrahedrons t_i , which are taken to be the triangulations of the initial and the final surface. The corresponding Feynman diagram will be given by the dual one-complex, i.e. a four-valent graph with 8 external legs. For the invariant amplitude one has to consider the $D = 3$ analog of (42), i.e. to a tetrahedron whose edges are labeled with group elements we assign

$$\Phi_4 = \int_{G^6} d^6 g \varphi(1, 2, 3) \varphi(1, 4, 5) \varphi(2, 5, 6) \varphi(3, 6, 4) \quad . \quad (69)$$

By expanding φ 's in the Fourier modes we obtain

$$\Phi_4 = \sum_{\Lambda, \alpha} \phi_{\alpha(123)}^{\Lambda(123)} \phi_{\alpha(145)}^{\Lambda(145)} \phi_{\alpha(256)}^{\Lambda(256)} \phi_{\alpha(364)}^{\Lambda(364)} \left\{ \begin{array}{ccc} \Lambda_1 & \Lambda_2 & \Lambda_3 \\ \Lambda_4 & \Lambda_5 & \Lambda_6 \end{array} \right\} . \quad (70)$$

Hence we define

$$\Phi_4(\Lambda_1, \dots, \Lambda_6) = \frac{1}{N(\Lambda)} \sum_{\alpha} \phi_{\alpha(123)}^{\Lambda(123)} \phi_{\alpha(145)}^{\Lambda(145)} \phi_{\alpha(256)}^{\Lambda(256)} \phi_{\alpha(364)}^{\Lambda(364)} \quad , \quad (71)$$

[§]This is also a face if Γ is considered as a two-complex. This two-complex is called a spin foam [11].

where a natural normalization would be $N(\Lambda) = \sqrt{d(\Lambda_1) \cdots d(\Lambda_6)}$.

We then associate the quantity $\Phi(\Lambda_1, \dots, \Lambda_6)$ to the spin net corresponding to a triangulation of S^2 given by a tetrahedron t . The spin net graph will be given by the dual one-complex to t , where t is considered as a 2d simplicial complex. One then obtains the Mercedes-Benz graph, and its edges will be labeled with the irreps Λ_i . Hence a perturbative amplitude for the transition from $|\Phi(\Lambda_1, \dots, \Lambda_6)\rangle$ to $|\Phi(\Lambda'_1, \dots, \Lambda'_6)\rangle$ will be given by

$$A_{if}(n) = \langle \Phi(\Lambda'_1, \dots, \Lambda'_6) (S_v)^n \Phi(\Lambda_1, \dots, \Lambda_6) \rangle, \quad (72)$$

where $n \geq 8$.

If we consider a $\mathcal{T}(S^2)$ which is two tetrahedrons with a common face, then the dual graph is a prism. If we attach to the edges of \mathcal{T} the group elements g_1, \dots, g_9 , we can then construct an invariant operator

$$\Psi_6 = \int_{G^9} d^9 g \varphi(1, 4, 2) \varphi(2, 5, 3) \varphi(1, 3, 6) \varphi(4, 7, 8) \varphi(5, 8, 9) \varphi(7, 6, 9) \quad (73)$$

By expanding into the Fourier modes we obtain $\Phi_6(\Lambda_1 \cdots \Lambda_9)$, which is an operator associated to the spin net given by a prism whose edges are labeled by the irreps $\Lambda_1 \cdots \Lambda_9$. In this way we obtain operators and states for 2d spin nets whose graphs are dual to triangulations of a boundary surface Σ . This surface represents a space part of the spacetime manifold M .

Therefore we denote the $\Phi_n(\Lambda)$ operators and states as $\Phi(\gamma, \Lambda)$, where γ denotes the spin net graph, which is dual to $\mathcal{T}(\Sigma)$, and Λ denotes the labeling of the edges of γ with the unitary irreps of G . One can write

$$\Phi(\gamma, \Lambda) = \frac{1}{N(\Lambda)} \sum_{\alpha} \prod_v \phi_{\alpha(e_1(v)e_2(v)e_3(v))}^{\Lambda(e_1(v)e_2(v)e_3(v))}, \quad (74)$$

where v are the vertices of γ , and $e_i(v)$ are the three edges coming out of the vertex v .

The amplitude (72) is equal to a sum of FD with n vertices and $6 + 6$ external legs. The value of the each diagram in that sum can be written as

$$A(\mathcal{T}) = N(\mathcal{T}) \sum_{\Lambda_{\tilde{e}}} 6j(\Lambda'_1 \cdots \Lambda'_6) \prod_{\tilde{e} \in \mathcal{T}} d(\Lambda_{\tilde{e}}) \prod_{\tilde{t} \in \mathcal{T}} 6j(\Lambda_{\tilde{e}_1(\tilde{t})} \cdots \Lambda_{\tilde{e}_6(\tilde{t})}) 6j(\Lambda_1 \cdots \Lambda_6), \quad (75)$$

where

$$N(\mathcal{T}) = \frac{d(\Lambda'_1) \cdots d(\Lambda'_6) d(\Lambda_1) \cdots d(\Lambda_6)}{N(\Lambda'_1 \cdots \Lambda'_6) N(\Lambda_1 \cdots \Lambda_6)} = \sqrt{d(\Lambda'_1) \cdots d(\Lambda'_6) d(\Lambda_1) \cdots d(\Lambda_6)}, \quad (76)$$

and \tilde{t} are the interior tetrahedra and \tilde{e} are the interior edges of a triangulation \mathcal{T} which is dual to the Feynman diagram, while the edges of the boundary tetrahedra are labeled with the irreps $\Lambda_1, \dots, \Lambda_6$ and $\Lambda'_1, \dots, \Lambda'_6$.

The amplitude $A(\mathcal{T})$ can be recognized as the Ponzano-Regge amplitude for the transition from an initial surface to a final one via triangulation \mathcal{T} of the spacetime

manifold whose boundaries are the initial and the final surface, where the edges of the initial and the final triangulations are colored with a fixed set of irreps [2, 10].

The general spin net amplitude

$$A_{if}(n) = \langle \Phi(\gamma', \Lambda') | (S_v)^n | \Phi(\gamma, \Lambda) \rangle , \quad (77)$$

can be computed from

$$A_{if}(n) = \langle \Phi(\gamma', \Lambda') (S_v)^n \Phi(\gamma, \Lambda) \rangle , \quad (78)$$

by using the formula (74). It will be given by the sum of all (v, v') FD with n vertices and $v + v'$ external legs, multiplied by the normalization factor $N^{-1}(\Lambda)N^{-1}(\Lambda')$, where v and v' are the numbers of vertices in the spin nets γ and γ' , respectively. After summing over the representation indices, each FD will give the Ponzano-Regge amplitude for the corresponding 3-complex \mathcal{T} consisting of n tetrahedrons and with two disjoint boundaries consisting of v and v' tetrahedrons. One can then write

$$A_{if}(n) = \sum_{\mathcal{T}, |\mathcal{T}|=n} A(\mathcal{T}; \gamma, \Lambda; \gamma', \Lambda') . \quad (79)$$

The colored boundary complexes \mathcal{T}_v and $\mathcal{T}_{v'}$ are dual to the spin nets γ and γ' since they can be considered as two-complexes corresponding to triangulations of the initial and the final surface.

Also, according to our time variable interpretation, the amplitude (79) will correspond to the transition from the initial to the final state in n units of the covariant time.

Note that the spin network states $|\Phi(\gamma, \Lambda)\rangle$ have the same labeling as the spin network states $|\Psi(\gamma, \Lambda)\rangle$ obtained by the canonical quantization of 3d gravity, or equivalently, the $SU(2)$ BF theory [11]. However, these states are fundamentally different, because they belong to different Hilbert states, which is the consequence of two different quantization procedures which are used.

In the canonical approach, the spacetime manifold M has a fixed topology $\mathbf{R} \times \Sigma$, where Σ is a 2d surface. In the connection representation, one can construct a Hilbert space of gauge invariant states $L_2(\mathcal{A}/\mathcal{G})$, whose orthonormal basis is given by the spin network states $\Psi_{\gamma, \Lambda}(A)$. These are evaluated by assigning the open holonomies $U^\Lambda(e, A)$ to the edges e of the spin net graph γ which is embedded in Σ , where Λ is the irrep assigned to the edge e . One then takes the product of the holonomy matrices and contracts the indices with the intertwiners in order to obtain a scalar wave-function $\Psi_{\gamma, \Lambda}(A)$.

This construction can be discretized [10, 11], where Σ is replaced by a triangulation $\mathcal{T}(\Sigma)$, and γ is the dual graph, while the connection is replaced by a set of group elements g_e representing the holonomies of the edges e , so that $U_{\alpha\beta}^\Lambda(e, A) \rightarrow D_{\alpha\beta}^\Lambda(g_e)$. Then Ψ becomes an element of the space $L_2(G^E)$ where E is the number of the edges of γ [14]. Hence in this formulation, one can consider only the spin nets with a fixed

number of the edges, so that one cannot calculate the transition amplitude between the spin nets having different numbers of edges. On the other hand, in the case of field theory spin networks, there is no such a constraint, because the Hilbert space has the Fock space structure, and one can have spin net states with different numbers of the edges. Hence the field theory Hilbert space contains states describing all colored 2-complexes, or equivalently, the states for manifolds Σ of all topologies.

Note that a single complex amplitude in (79) can be written as a scalar product of the canonical quantization spin net states $|\Psi(\gamma, \Lambda)\rangle$ as

$$A(\mathcal{T}, \gamma, \Lambda; \gamma', \Lambda') = \langle \Psi(\gamma', \Lambda') | \Psi(\gamma, \Lambda) \rangle, \quad (80)$$

when γ and γ' have the same numbers of edges [15, 16]. On the other hand, it follows from (79) that

$$\begin{aligned} A_{if}(n) &= C(n, \gamma, \gamma') \langle \Psi(\gamma', \Lambda') | \Psi(\gamma, \Lambda) \rangle \\ &= \langle \Phi(\gamma', \Lambda') | (S_v)^n | \Phi(\gamma, \Lambda) \rangle, \end{aligned} \quad (81)$$

where $C(n, \gamma, \gamma')$ is the number of 3-complexes \mathcal{T} with n tetrahedrons whose boundary complexes are dual to γ and γ' , or equivalently, it is the number of (v, v') FD with n vertices. Hence due to the topological nature of 3d gravity, one can relate the canonical and the field theory spin net states.

7 D=4 model

The case of four spacetime dimensions is special for the BF theory, since this is the dimension where the BF theory starts to be different from the GR theory. GR in $D \geq 4$ is not a topological theory, or in other words, it has local degrees of freedom. This is also reflected by the fact that the two-form B_{ab} , where $B = B^{ab} J_{ab}$ and J_{ab} are the $SO(4)$ or $SO(3, 1)$ Lie algebra generators, cannot be always written as $e^a \wedge e^b$, where e^a is the vierbein one-form. Recall that in 3d $B^{ab} = \epsilon^{abc} e_c$, so that the BF action becomes identical to the Palatini form of the Einstein-Hilbert action. Therefore in order to obtain a simplicial non-topological gravity theory from the BF theory, something has to be modified in a 4d analog of the constructions described in the previous sections.

The field theory which generates the partition function of the topological gravity theory in 4d was worked out by Ooguri [6]. The labeling pattern in the 4d action is a straightforward generalization of the $D = 2, 3$ theories. We assign $\varphi(g_1, \dots, g_4)$ to a tetrahedron whose faces are labeled with the group elements g_i , and the interaction term is given by a 4-simplex built out of 5 tetrahedrons, so that

$$\begin{aligned} S_4 &= \frac{1}{2} \int_{G^4} d^4 g \varphi^2(1, 2, 3, 4) \\ &+ \frac{\lambda}{5!} \int_{G^{10}} d^{10} g \varphi(1, 2, 3, 4) \varphi(4, 5, 6, 7) \varphi(7, 3, 8, 9) \varphi(9, 6, 2, 10) \varphi(10, 8, 5, 1). \end{aligned} \quad (82)$$

Since φ has to be G -invariant, we have

$$\varphi(g_1, g_2, g_3, g_4) = \int_G dg \phi(g_1 g, g_2 g, g_3 g, g_4 g) \equiv P_G \phi(g_1, g_2, g_3, g_4) \quad . \quad (83)$$

This gives the following momentum mode expansion

$$\varphi = \sum_{\Lambda, \Lambda', \alpha} \sqrt{d(\Lambda_1) d(\Lambda_2) d(\Lambda_3) d(\Lambda_4)} \phi_{\alpha(1234)}^{\Lambda(1234)\Lambda'} D_{\alpha(1234)}^{\Lambda(1234)\Lambda'}(g_1, g_2, g_3, g_4) \quad , \quad (84)$$

where

$$\phi_{\alpha(1234)}^{\Lambda(1234)\Lambda'} = \sum_{\beta} \phi_{\alpha\beta(1234)}^{\Lambda(1234)} \bar{C}_{\beta(1234)}^{\Lambda(1234)\Lambda'} \quad , \quad (85)$$

and

$$D_{\alpha(1234)}^{\Lambda(1234)\Lambda'}(g_1, g_2, g_3, g_4) = \sum_{\beta} \prod_{i=1}^4 D_{\alpha_i \beta_i}^{\Lambda_i}(g_i) C_{\beta(1234)}^{\Lambda(1234)\Lambda'} \quad . \quad (86)$$

The coefficients C are components of the intertwiner map ι_4

$$\iota_4 : V(\Lambda_1) \otimes V(\Lambda_2) \otimes V(\Lambda_3) \otimes V(\Lambda_4) \rightarrow V(0) \quad , \quad (87)$$

in an orthonormal basis.

In order to obtain (84) one needs the relation

$$\int_G dg U^{\Lambda_1}(g) \otimes U^{\Lambda_2}(g) \otimes U^{\Lambda_3}(g) \otimes U^{\Lambda_4}(g) = \iota_4 \iota_4^\dagger \quad , \quad (88)$$

which in an orthonormal basis $|\alpha\rangle$ becomes

$$\int_G dg D_{\alpha_1 \beta_1}^{\Lambda_1}(g) D_{\alpha_2 \beta_2}^{\Lambda_2}(g) D_{\alpha_3 \beta_3}^{\Lambda_3}(g) D_{\alpha_4 \beta_4}^{\Lambda_4}(g) = \sum_{\Lambda'} C_{\alpha(1234)}^{\Lambda(1234)\Lambda'} C_{\beta(1234)}^{\Lambda(1234)\Lambda'} \quad , \quad (89)$$

where Λ' labels the singlets in the tensor product of four irreps $\Lambda_1, \dots, \Lambda_4$.

By inserting (84) into S_4 we obtain for the kinetic part

$$S_k = \frac{1}{2} \sum_{\Lambda, \Lambda', \alpha} \left| \phi_{\alpha(1234)}^{\Lambda(1234)\Lambda'} \right|^2 \quad , \quad (90)$$

while for the interaction part we get

$$S_v = \frac{\lambda}{5!} \sum_{\Lambda, \Lambda', \alpha} \phi_{\alpha(1234)}^{\Lambda(1234)\Lambda'_1} \phi_{\alpha(4567)}^{\Lambda(4567)\Lambda'_2} \phi_{\alpha(7389)}^{\Lambda(7389)\Lambda'_3} \phi_{\alpha(96210)}^{\Lambda(96210)\Lambda'_4} \phi_{\alpha(10851)}^{\Lambda(10851)\Lambda'_5} \left\{ \begin{matrix} \Lambda_1 & \cdots & \Lambda_{10} \\ \Lambda'_1 & \cdots & \Lambda'_5 \end{matrix} \right\} \quad , \quad (91)$$

where the 15j symbol is defined as

$$\left\{ \begin{matrix} \Lambda_1 & \cdots & \Lambda_{10} \\ \Lambda'_1 & \cdots & \Lambda'_5 \end{matrix} \right\} = \sum_{\alpha} C_{\alpha(1234)}^{\Lambda(1234)\Lambda'_1} C_{\alpha(4567)}^{\Lambda(4567)\Lambda'_2} C_{\alpha(7389)}^{\Lambda(7389)\Lambda'_3} C_{\alpha(96210)}^{\Lambda(96210)\Lambda'_4} C_{\alpha(10851)}^{\Lambda(10851)\Lambda'_5} \quad . \quad (92)$$

We have written the formulas (91) and (92) for real $D(g)$ matrices. In the complex case the repeated α indices will have a minus sign and there will be the sign factors coming from the complex conjugation rules for the $D(g)$ matrix elements. Since this will not be important for our purposes, we will use the simpler real basis notation.

The kinetic action (90) implies for the propagator

$$\langle \phi_{\alpha(1234)}^{\Lambda(1234)\Lambda} \phi_{\alpha'(1234)}^{\Lambda'(1234)\Lambda'} \rangle = \delta^{\Lambda, \Lambda'} \prod_{i=1}^4 \delta^{\Lambda_i, \Lambda'_i} \delta_{\alpha_i, \alpha'_i} . \quad (93)$$

Note that for a tetrahedron with labeled faces it is natural to consider φ which is invariant under the permutations of the labels. This would then give the constraints on the Fourier modes (the analog of the hermiticity in 2d case), which would involve the $6j$ symbols [6]. As a result, the propagator will not have the delta function for the intertwiner labels, but it will have the $6j$ symbol for Λ_i, Λ and Λ' . Since it is not clear whether the symmetric φ is necessary, and in order to make the presentation simpler, we will not impose the permutation symmetry.

The Feynman diagrams will be given by the 5-valent graphs. These can be represented as ribbon graphs, where each edge is replaced with four parallel lines, and each vertex is replaced with a pentagram whose corners are cut, and the incident lines for a corner are joined with the four lines of an edge. The value of a FD can be obtained by coloring the ten lines of each pentagram with ten irreps Λ , as well as every corner with the intertwiner label Λ' . One then assigns to each pentagram the corresponding $15j$ symbol. For each propagator line one takes a delta function of the labels it connects. One then takes the product of all delta functions with the $15j$ symbols, and sums over the labels.

Consider a vacuum FD Γ with $|V|$ vertices, which is a dual one complex to a 4d simplicial complex \mathcal{T} consisting of $|V|$ 4-simplices. For example, the hexagon graph, a six vertex 5-valent graph with all the vertices connected, is a dual one-complex for a 5-simplex considered as a 4d simplicial complex. This 4d complex can be considered as a triangulation \mathcal{T} of S^4 , which consists of five 4-simplices. Since each closed propagator line gives a $d(\Lambda)$ factor, the Feynman rules associate to each loop (face) F of Γ an irrep Λ_F with the weight $d(\Lambda_F)$. For each vertex there is the $15j$ symbol of 10 loops (faces) which share it plus 5 intertwiner labels of the 5 edges coming out of the vertex. Hence one obtains

$$Z(\Gamma) = \sum_{\Lambda_F, \Lambda'_E} \prod_{F \in \Gamma} d(\Lambda_F) \prod_{V \in \Gamma} \left\{ \begin{array}{ccc} \Lambda_{F_1(V)} & \cdots & \Lambda_{F_{10}(V)} \\ \Lambda'_{E_1(V)} & \cdots & \Lambda'_{E_5(V)} \end{array} \right\} . \quad (94)$$

Since a triangle f of \mathcal{T} is dual to a face F of Γ , a tetrahedron t to the edge E , and a four-simplex s to a vertex V , we can rewrite the above expression as

$$Z(M) = \sum_{\Lambda_f, \Lambda'_t} \prod_{f \in \mathcal{T}} d(\Lambda_f) \prod_{s \in \mathcal{T}} \left\{ \begin{array}{ccc} \Lambda_{f_1(s)} & \cdots & \Lambda_{f_{10}(s)} \\ \Lambda'_{t_1(s)} & \cdots & \Lambda'_{t_5(s)} \end{array} \right\} . \quad (95)$$

Hence $Z(M)$ is a sum over colorings of the complex $\mathcal{T}(M)$, where each triangle f of \mathcal{T} is colored with an irrep Λ_f , while each tetrahedron t of \mathcal{T} is colored with an intertwiner label Λ'_t from the tensor product of four irreps coloring the triangles of t . This sum is divergent, and by going to the quantum group formalism one obtains a finite number which is independent of the triangulation, and hence represents a topological invariant of the 4-manifold [4].

The form (95) of the partition function is the state sum which mathematicians have explored over the years, and it was in this approach that the modification which lead to quantum GR was discovered [8]. Namely, the constraint $B^{ab} = e^a \wedge e^b$, which transforms the BF action into Palatini action, can be written in a simple form as

$$B^{ab} \wedge B_{ab} = 0 \quad , \quad (96)$$

where the indices are contracted with the group metric. The constraint (96) is translated in the state-sum formalism into a constraint on the representations with which one colors the triangles of a triangulation as

$$J^{ab} J_{ab} = 0 \quad , \quad (97)$$

where J^{ab} are the generators of the $SO(4)$ Lie algebra $so(4)$. Since $so(4) = su(2) \oplus su(2)$, the constraint (97) becomes

$$\vec{J}^2 - \vec{K}^2 = 0 \quad , \quad (98)$$

where \vec{J}^2 and \vec{K}^2 are the Casimirs of the two $SU(2)$.

In terms of representations this constraint implies that one labels the triangles f of a 4-simplex s with the simple irreps $N = (j, j)$, and instead of summing over all irreps $\Lambda = (j, k)$ in the state sum, one sums only over the simple irreps. Then the $15j$ symbol becomes a sort of $10j$ symbol, the Barret-Crane vertex

$$\mathcal{V}(N_1 \cdots N_{10}) = \sum_{M, \alpha} C_{\alpha(1234)}^{N(1234)M_1} C_{\alpha(4567)}^{N(4567)M_2} C_{\alpha(7389)}^{N(7389)M_3} C_{\alpha(96210)}^{N(96210)M_4} C_{\alpha(10851)}^{N(10851)M_5} \quad , \quad (99)$$

where M labels the simple irrep singlets appearing in the tensor product of four simple irreps. Hence

$$Z_{BC}(\mathcal{T}) = \sum_{N_f} \prod_{f \in \mathcal{T}} d(N_f) \prod_{s \in \mathcal{T}} \mathcal{V}(N_{f_1(s)} \cdots N_{f_{10}(s)}) \quad , \quad (100)$$

where we have written Z as a function of \mathcal{T} , since now the state sum will depend on the local parameters of the triangulation \mathcal{T} .

An important property of the BC vertex is that it can be expressed in terms of an integral over $SU(2)^5$ [17]

$$\mathcal{V}(N_1 \cdots N_{10}) = \int_{SU(2)^5} d^5 h \prod_{1 \leq k < l \leq 5} Tr_{j_{kl}}(h_k h_l^{-1}) \quad , \quad (101)$$

where $(j_{12}, \dots, j_{45}) = (N_1, \dots, N_{10})$ label the edges of a 4-simplex s , and h_i label the vertices. This formula can be easily proved by noticing that the right-hand expression is $[10j(SU(2))]^2$ and since $SO(4) \simeq SU(2) \times SU(2)$, then $10j(SO(4))|_{j=\text{simple}} = [10j(SU(2))]^2$.

The explanation of this formula comes from the fact that

$$SU(2) \simeq S^3 = SO(4)/SO(3) \quad ,$$

so that the h variables are really coordinates x on the homogeneous space $X = G/H$, which is a consequence of the fact that $N = (j, j)$ is a class-one representation of $G = SO(4)$ with respect to the subgroup $H = SO(3)$ [18]. The quantity $Tr_j(h_k h_l^{-1})$ is the zonal spherical function, and can be thought of as a Green's function on the X space, so that the BC vertex can be represented as a Feynman graph of a field theory which lives on X . One then obtains

$$G_N(x_1, x_2) = Tr_j(h_1 h_2^{-1}) \quad (102)$$

$$= \frac{\sin((2j+1)\theta_{12})}{\sin \theta_{12}} \quad , \quad (103)$$

where θ_{12} is the angle between the unit 4-vectors x_i , or equivalently, the geodesic distance between the points of X . The vectors x_i can be interpreted as the normals to the tetrahedrons t_i which share the face f_{12} labeled with the irrep j .

The representation (101) combined with the formula (103) gives the asymptotics of \mathcal{V} in the limit of large j

$$\mathcal{V} \sim \sum_{\mathcal{A}} \Delta(A) \cos \left(\sum_{f \in s} A_f \theta_f + \pi/4 \right) \quad , \quad (104)$$

where $A_f = 2j_f + 1$ is the area of the face f , and one has to sum over certain inequivalent colorings \mathcal{A} of the four-simplex s , for details see [19]. When (104) is inserted in Z_{BC} , it is easy to see that the terms of the form Qe^{iS_R} will appear, where $S_R = \sum_{f \in \mathcal{T}} A_f \delta_f$ is the Regge form of the EH action. This is an indication that the semi-classical limit of the BC model is a theory that is related to classical GR. However, the semi-classical limit of the BC model has to be investigated in more detail in order to be able to make more definite statements.

Another important consequence of the formula (101) is that it can be easily generalized to the Lorentzian case [9], since then $G = SO(3, 1) \simeq SL(2, C)$, $H = SO(3)$ and therefore X is a hyperboloid. The unitary irreps of $SL(2, C)$ are labeled with two numbers, i.e. $\Lambda = (n, p)$ where $n \in \mathbf{Z}$ and $p \in \mathbf{R}$, and the simple representations are $(0, p)$ and $(n, 0)$. It is natural to assign $(0, p)$ irreps to space-like triangles and $(n, 0)$ to time-like triangles. One can restrict the model to the continuous irreps only so that

$$G_p(x_1, x_2) = \frac{\sin(p d_{12})}{p \sinh d_{12}} \quad , \quad (105)$$

where d_{12} is the geodesic distance between x_1 and x_2 . By using (105) one can construct the Lorentzian analog of the BC vertex via

$$\mathcal{V}(N_1 \cdots N_{10}) = \int_{X^5} d^5x \prod_{1 \leq k < l \leq 5} G_{p_{kl}}(x_k, x_l) \quad , \quad (106)$$

where $(N_1, \dots, N_{10}) = (p_{12}, \dots, p_{45})$.

As discussed in [18] for the Euclidean case and in [9] for the Lorentzian case, the constructions (101) and (106) can be generalized to give expressions for higher nj symbols. Given a graph γ we can label the vertices of γ with the coordinates x_k from the homogeneous space $X = G/H$. Then associate to each edge (k, l) of γ a simple irrep labeled with j_{kl} . In this way one obtains a labeled graph which is called a relativistic spin net. We can then associate the “propagator” $G_{j_{kl}}(x_k, x_l)$ to each edge, and if we now consider γ as a Feynman diagram of a theory with delta function vertices, then its value will be given by

$$\mathcal{V}(N_1 \cdots N_n) = \int_{X^m} d^m x \prod_{1 \leq k < l \leq m} G_{j_{kl}}(x_k, x_l) \quad , \quad (107)$$

where $\{j_{kl} | 1 \leq k < l \leq m\} = \{N_1, \dots, N_n\}$.

In the Euclidean case, the expression (107) is finite, since it is equal to a finite sum of products of a finite number of C symbols. This is a consequence of the formula (102). In the Lorentzian case, the finiteness is not obvious. The $10j$ symbol was conjectured to be finite in [9], and recently a proof of finiteness was given in [20].

As far as the finiteness of the state sum $Z_{BC}(\mathcal{T})$ is concerned, it is divergent, but as argued in [9], one expects to become finite in the quantum group formalism. It is interesting that there is a modification of the BC field theory model in the Euclidean case which seems to give finite Z for all \mathcal{T} [21, 23].

8 BC field theory

Now we can discuss the field theory formulation of the BC model [12]. The constraint of using only the simple representations to label the triangles of a triangulation is translated in the field theory formalism into a requirement of invariance under the $SO(3)$ subgroup: $\varphi(g_i) = \varphi(g_i h_i)$, $h_i \in H$. Hence

$$\varphi(g_1, g_2, g_3, g_4) = \int_{H^4} d^4h \phi(g_1 h_1, g_2 h_2, g_3 h_3, g_4 h_4) \equiv P_H \phi \quad , \quad (108)$$

where ϕ is an unconstrained field. By combining the H -invariance (108) with the usual G -invariance (83) one obtains

$$\begin{aligned} \varphi &= P_G P_H \phi = \int_G dg \int_{H^4} d^4h \phi(g_i g h_i) \\ &= \sum_{N, \alpha, \beta} \sqrt{d(N_1) \cdots d(N_4)} \phi_{\alpha(1234)}^{N(1234)} \prod_{i=1}^4 D_{\alpha_i \beta_i}^{N_i}(g_i) B_{\beta(1234)}^{N(1234)} \quad , \end{aligned} \quad (109)$$

where

$$B_{\beta(1234)}^{N(1234)} = \frac{1}{\sqrt{\Delta(1234)}} \sum_M C_{\beta(1234)}^{N(1234)M}. \quad (110)$$

M denotes the simple irreps whose singlets appear in the tensor product of four simple irreps, and Δ is the dimension of that subspace. By inserting (109) into S_4 one obtains

$$S_k = \frac{1}{2} \sum_{N,\alpha} \left| \phi_{\alpha(1234)}^{N(1234)} \right|^2, \quad (111)$$

while for the interaction part we get

$$S_v = \frac{\lambda}{5!} \sum_{N,\alpha} \phi_{\alpha(1234)}^{N(1234)} \phi_{\alpha(4567)}^{N(4567)} \phi_{\alpha(7389)}^{N(7389)} \phi_{\alpha(96210)}^{N(96210)} \phi_{\alpha(10851)}^{N(10851)} \tilde{\mathcal{V}}(N_1 \cdots N_{10}), \quad (112)$$

where $\tilde{\mathcal{V}}$ is the normalized BC vertex

$$\tilde{\mathcal{V}} = \frac{\mathcal{V}(N_1 \cdots N_{10})}{\sqrt{\Delta(1234) \cdots \Delta(10851)}}. \quad (113)$$

Hence the FD of the field theory defined by (112) will be given by the closed and open 5-valent graphs, as in the topological case, but the propagator and the vertex factors will be different. A vacuum FD Γ will reproduce the state sum (100) for a triangulation \mathcal{T} whose dual one-complex is Γ . However, now one expects that the regularized sum will depend on the triangulation \mathcal{T} , since $Z(\mathcal{T})$ does not have the properties of a topological invariant.

Note that there exists a modification of the BC field theory action which gives finite FD in the Euclidean case without using the quantum group as a regulator [21, 23]. The modified action has the same kinetic term as the topological action (G invariance only), while the interaction term is the same as in the BC case (G and H invariance). When expanded in the Fourier modes, one obtains

$$S_k = \frac{1}{2} \sum_{N,M,\alpha} (d_{N_1} \cdots d_{N_4})^2 \left| \phi_{\alpha(1234)}^{N(1234)M} \right|^2 \quad (114)$$

for the kinetic term, while

$$S_v = \frac{\lambda}{5!} \sum_{N,M,\alpha} \phi_{\alpha(1234)}^{N(1234)M_1} \phi_{\alpha(4567)}^{N(4567)M_2} \phi_{\alpha(7389)}^{N(7389)M_3} \phi_{\alpha(96210)}^{N(96210)M_4} \phi_{\alpha(10851)}^{N(10851)M_5} \tilde{\mathcal{V}}(N_1 \cdots N_{10}), \quad (115)$$

is the interaction term. The appearance of the factor $(d_{N_1} \cdots d_{N_4})^2$ in the kinetic term gives a damping factor $(d_{N_1} \cdots d_{N_4})^{-2}$ in the propagator which insures the finiteness.

As far as the FD with external legs are concerned, they will correspond to triangulations of a 4-manifold with boundaries, where each boundary contains more than

one 4-simplex. The boundary complexes are then interpreted as 3d complexes, corresponding to triangulations of boundary 3-manifolds Σ_i . These FD will be relevant for the transition amplitude from a state on a 3-manifold Σ_1 to a state on a 3-manifold Σ_2 , where $\partial M = \Sigma_1 \cup \Sigma_2$. However, we need to construct first the spin net state associated to a triangulation of a 3-manifold Σ representing a boundary of M . This will be given by a straightforward generalization of the results from the lower dimensions.

Consider a 4-valent connected closed graph γ which corresponds to a triangulation of Σ , i.e. γ is a dual one complex to $\mathcal{T}(\Sigma)$, where now $\mathcal{T}(\Sigma)$ is considered as a 3d simplicial complex. For example, a 4-simplex is dual to a pentagram. A more complicated example is a 3d simplicial complex consisting of 8 tetrahedrons. The dual graph will have eight 4-valent vertices and 16 edges.

Given a triangulation $\mathcal{T}(\Sigma)$ we can construct a functional Φ by integrating a product of $\varphi_i(g_j)$ over G^n , where n is the number of triangles in $\mathcal{T}(\Sigma)$ and $\varphi_i(g_j)$ is associated to the tetrahedron in $\mathcal{T}(\Sigma)$ whose triangles are colored with the group elements g_j , i.e.

$$\Phi = \int_{G^n} d^n g \prod_{t \in \mathcal{T}(\Sigma)} \varphi(g_{f_1(t)}, g_{f_2(t)}, g_{f_3(t)}, g_{f_4(t)}) \quad . \quad (116)$$

By Fourier expanding it, we get

$$\Phi = \sum_{N, \alpha} \phi_{\alpha(1234)}^{N(1234)} \cdots \phi_{\alpha(n_1 n_2 n_3 n_4)}^{N(n_1 n_2 n_3 n_4)} \mathcal{V}_{\gamma}(N_1 N_2 \cdots N_{n_4}) , \quad (117)$$

where the vertex $\mathcal{V}_{\gamma}(N_1 \cdots N_{n_4})$ is given by (107) and γ is the four-valent graph associated to $\mathcal{T}(\Sigma)$, i.e. a dual one-complex to $\mathcal{T}(\Sigma)$. Hence we take the operator

$$\begin{aligned} \Phi(\gamma, N) &= \frac{1}{N_0(N)} \sum_{\alpha} \phi_{\alpha(1234)}^{N(1234)} \cdots \phi_{\alpha(n_1 n_2 n_3 n_4)}^{N(n_1 n_2 n_3 n_4)} \\ &= \frac{1}{N_0(N)} \sum_{\alpha} \prod_{v \in \gamma} \phi_{\alpha(e_1(v) \dots e_4(v))}^{N(e_1(v) \dots e_4(v))} \end{aligned} \quad (118)$$

to represent the spin net given by the graph γ whose edges are labeled with the simple irreps N_1, \dots, N_{n_4} . In the case of the Perez-Rovelli model, one obtains

$$\Phi(\gamma, N) = \frac{1}{N_0(N)} \sum_{M, \alpha} \prod_{v \in \gamma} \phi_{\alpha(e_1(v) \dots e_4(v))}^{N(e_1(v) \dots e_4(v)) M_v} . \quad (119)$$

The amplitude

$$A_{12}(n) = \langle \Phi(\gamma_2, N_2) (S_v)^n \Phi(\gamma_1, N_1) \rangle \quad , \quad (120)$$

will be interpreted as the amplitude for a transition from the state $|\Phi(\gamma_1, N_1)\rangle$ to the state $|\Phi(\gamma_2, N_2)\rangle$ in n units of the covariant time T . This will be a finite sum of FD, consisting of (v_1, v_2) FD with n vertices and $v_1 + v_2$ external legs, where v_i is the

number of the vertices in the spin net γ_i . Each FD would give a 4d analog of the single-complex Ponzano-Regge amplitude, if we choose a normalization $N_0(N) = \sqrt{\prod_k d(N_k)}$.

Note that the amplitude (120) will be finite in the Euclidean Perez-Rovelli model. Therefore this would be an example of a finite 4d quantum gravity theory with local degrees of freedom, provided that one uses the covariant time interpretation. In the Lorentzian case it is not known whether the Perez-Rovelli model is finite, but it is plausible that it may be finite. Alternatively, one can try the quantum Lorentz group regularization.

As in the 3d case, the field theory spin nets Φ can be considered as generalizations of the canonical quantization spin nets Ψ [24] to situations when the topology of the spacetime manifold is not fixed. Also, due to the non-topological nature of 4d gravity, the labels are not the same anymore. Covariant spin nets have simple $SO(4)$ or $SL(2, C)$ irreps as labels, while the canonical spin nets have the $SU(2)$ labels. In order to make a closer link between these, one would have to discretize the canonical spin nets, as in the 3d case, but in the 4d case one would have to solve the diffeomorphism and the Hamiltonian constraints. Furthermore, one would have to find a time variable in order to compute the transition amplitudes. A great advantage of the covariant approach, which is related to the fact that it's a path-integral quantization, is that one does not have to do all these non-trivial steps, and one can calculate directly the transition amplitudes.

9 Conclusions

We have demonstrated that there is a significant benefit if one completes the Feynman diagram picture of the transition amplitudes for the state-sum models of quantum gravity by introducing the Fourier mode operators and the corresponding Fock space, so that the transition amplitudes can be understood as matrix elements of an evolution operator, in exact analogy with the particle field theory case. The spin network operators are constructed from the products of the field Fourier modes, such that each mode corresponds to a vertex of the spin net graph, and then one averages over the spin components in a group-theory invariant way. The corresponding state can be thought of as a state of a universe whose space is discrete. This 3d space is given by the dual 3-complex to the spin net graph γ , and the quantum states are determined by the colorings of the triangles with the irreps Λ .

We have considered only the transition amplitudes for the single-universe states. Since one can act on the Fock space vacuum with more than one spin net operator, one can obtain the multi-universe states, and the corresponding transition amplitudes. Hence our quantum field theory of spin networks is also a quantum field theory of discrete universes. This QFT is then a discrete realization of the third quantization of gravity.

Note that the expressions for the amplitudes we have written are mostly formal,

and they need a regularization for the infinite sums over the irreps of G . The standard approach is to use the quantum group formalism, which is a group invariant way of putting a cut off on Λ . Alternatively, the Euclidean BC model can be regularized via Perez-Rovelli reformulation. Hence in this model the amplitudes for finite time intervals will be finite. The same could apply for the Lorentzian version of the Perez-Rovelli model [22], provided one proves the finiteness of an arbitrary Feynman diagram. If this turns out to be true, one would have a well defined quantum theory of 4d Lorentzian gravity. Then the main task would be to examine the semi-classical states of this theory. Given the discrete nature of the theory, one expects that such states would be described by spin nets of large number of vertices, since large triangulations represent good approximations of smooth manifolds.

The key ingredient in making the quantum field theory of spin nets well defined is the introduction of the discrete time variable T and the corresponding evolution operator $U(T)$. Since $U(T) = (S_v)^T$ one can define the transition amplitudes for finite time intervals, which are given by the sum of FD with T vertices, or equivalently, by the sum of partition functions for triangulations with T simplices. In this way one avoids the infinite sum of the single-complex amplitudes, or Feynman diagrams, for all possible complexes interpolating between the boundary complexes, which is often invoked in the literature as the physical amplitude. We have demonstrated that this is not necessary, since this prescription is related to the evolution operator $U = \exp(iS_v)$, and this operator is not natural in the covariant quantization formulation of theories without a fixed spacetime background.

The construction of the covariant time evolution operator mirrors the physical intuition that summing over all possible triangulations of the spacetime manifold corresponds to including metrics which give arbitrary large geodesic distance between the initial and the final boundary. Therefore one should divide this sum into parts corresponding to fixed time-like geodesic lengths. These partial sums still contain infinitely many triangulations. Hence a better time variable, i.e. a variable which gives finite partial sums, is the discretized analog of the spacetime volume, which is the number of the D -simplices in a simplicial complex representing the spacetime. Also, this interpretation is natural for the compact universes.

In order to gain a better understanding of the covariant time, one would have to see what is its relation to canonical time variables[¶], as well as a relation to matter clock variables. This analysis will be also important for understanding the meaning of the non-unitarity of the covariant time evolution operator.

If one takes the covariant evolution operator as the fundamental object, this would mean abandoning the law of unitary evolution of states in quantum mechanics (QM). However, this may not be a bad thing in this context, since we are dealing with the wavefunction of the Universe, which as a concept is not well defined in the standard

[¶]These are time variables in the canonical formulation of GR [25]. In this context a typical canonical time variable would be the volume of a spatial section, which would correspond to the number of vertices in the spin net.

QM due to the fact that there is no outside observer. Also, it could provide a mechanism for the the matter wavefunction collapse, which would be a concrete realization of gravity driven wavefunction collapse [26]. Also note that the QFT formulation of spin nets is also a good arena for exploring other interpretation schemes of QM for quantum gravity [25].

In order to explore all these ideas, as well as in order to have a complete theory, one would need a simplicial field theory formulation of matter coupled to gravity.

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References

- [1] T. Regge, *Nuovo Cimento* 19 (1961) 558
- [2] G. Ponzano and T. Regge, in *Spectroscopy and Group Theoretical Methods in Physics*, eds F. Block et al, North Holland, Amsterdam (1968)
- [3] V.G. Turaev and O.Y. Viro, *Topology* 31 (1992) 865
- [4] L. Crane and D.N. Yetter, in *Quantum Topology*, eds L.H. Kauffman and R. Baadhio, World Scientific, Singapore (1993), hep-th/9301062;
L. Crane, D.H. Kauffman and D.N. Yetter, preprint hep-th/9409167
- [5] D.V. Boulatov, *Mod. Phys. Lett. A* 7 (1992) 1629
- [6] H. Ooguri, *Mod. Phys. Lett. A* 7 (1992) 2799
- [7] F. David, *Simplicial quantum gravity and random lattices*, Les Houches Lectures, July 1992, hep-th/9303127
- [8] J.W. Barrett and L. Crane, *J. Math. Phys.* 39 (1998) 3296
- [9] J.W. Barrett and L. Crane, *Class. Quant. Grav.* 17 (2000) 3101
- [10] L. Freidel, *Nucl. Phys. B (Proc. Suppl.)* 88 (2000) 237
- [11] J. Baez, *An introduction to spin foam models of BF theory and quantum gravity*, preprint gr-qc/9905087
- [12] R. De Pietri, L. Freidel, K. Krasnov and C. Rovelli, *Nucl. Phys. B* 574 (2000) 785

- [13] R. Jackiw, in Quantum Theory of Gravity, ed S.M. Christiansen, Adam Hilger, Bristol (1984)
C. Teitelboim, in Quantum Theory of Gravity, ed S.M. Christiansen, Adam Hilger, Bristol (1984)
- [14] A. Ashtekar and J. Lewandowski, J. Diff. Geom. 17 (1995) 191
- [15] H. Ooguri, Nucl. Phys. B 382 (1992) 276
- [16] C. Rovelli, Phys. Rev. D 48 (1993) 2702
- [17] J.W. Barrett, Adv. Theor. Math. Phys. 2 (1998) 593
- [18] L. Freidel and K. Krasnov, J. Math. Phys. 41 (2000) 1681
- [19] J.W. Barrett and R.M. Williams, Adv. Theor. Math. Phys. 3 (1999) 1
- [20] J. Baez and J.W. Barrett, preprint gr-qc/0101107
- [21] A. Perez and C. Rovelli, preprint gr-qc/0006107
- [22] A. Perez and C. Rovelli, preprint gr-qc/0009021
- [23] A. Perez, preprint gr-qc/0011058
- [24] C. Rovelli and L. Smolin, Phys. Rev. D 52 (1995) 5743
- [25] C.J. Isham, Canonical quantum gravity and the problem of time, Nato Summer School lectures, Salamanca, June 1992, gr-qc/9210011
- [26] R. Penrose, in Quantum Concepts of Space and Time, eds R. Penrose and C.J. Isham, Clarendon Press, Oxford (1986)